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The Relationship Between Using Conceptual Language and the Depth of Student Understanding of Dynamic Addition and Multiplication in 4-9-Year-Old Montessori Students

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Abstract

This study aims to bring clarity to the relationship between procedural mathematical work and abstracted math learning when carrying in addition and multiplication. To explore this relationship, researchers employed both quantitative and qualitative data tools that unearthed the nuances within this specific process of math learning. Participants in the study included twenty-nine students from two different schools in different mixed age groups including ages three-to-six-years-old and six-to-nine-years-old. Students participated in a six-week intervention process, working on dynamic addition and multiplication using conceptual mathematical language to support the process. The findings indicate an overall two-point increase across learning variables post intervention. The conclusion of this study implores the broader educational community to revisit systemic, procedural math learning processes. In the future, we must question the finality of manipulatives and their place in the continuum of authentic math learning.

Keywords: math, manipulatives, carrying, regrouping, addition, multiplication, golden beads, Montessori math, number sense

Math describes fundamental aspects of the observable world. As such, mathematics in its more complex forms reveals the relationships between sophisticated ideas, which can lead to potential innovations and societal solutions of great value to humanity. The great mathematicians who developed buildings, infrastructure, machines, and technology used mathematics as a language which helped them understand and improve their world. Naturally, the response to these innovations was an appreciation for the way mathematics improved our quality of life. Thus, society has continued to emphasize mathematics heavily in education.

Math in the modern age, however, has sometimes been intermittently relegated to a subject of rote learning for the masses. Mathematics can be perceived as impenetrable, like an overly complicated game of chess existing on an invisible board which only mathematicians and scientists engage with. Math learning is not generally thought of as expressive, accessible, or enjoyable. Many reformers (Holm & Kajander, 2012; Roselli, 2018) seek to end this stale notion of mathematics and replace it with one that envisions mathematics as a creative endeavor rooted in real, tangible processes.

Cultivating creative thinking about mathematics begins with the foundation of early mathematics learning in preschool and elementary school. All too often, early mathematics learning across pedagogies relies on rote memorization and repetition of simple math concepts. The conversation surrounding the improvement of early mathematics instruction often includes material choice, learning objectives, and the layering of compounding mathematical knowledge. However, none of these components address real-world application or conceptual articulation. In the child’s early development from ages three to nine, math should introduce formal logical thought. It should grow and hone recognition of patterns and stepwise thinking that develops habits of mind which allow young people to become rational thinkers. Early math learning should usher students toward an understanding of the fundamental order of all things.

Consequently, robust early math learning is critical to later math success. Students in preschool and elementary classrooms are building mental frameworks that will support their later understanding of higher math. If a student's knowledge is merely rote or procedural, they may struggle to construct abstract connections. An observable struggle in Montessori math education is the progression from concrete repetition to abstraction. Montessori children use manipulatives which isolate and emphasize a specific quality. The use of concrete materials has proven helpful with initial learning. It is observable for some, however, that the transfer of concrete knowledge to a novel or abstracted problem is lacking.

Further clarification about how to help students deepen their understanding of mathematics by making consistent and efficient connections between concrete experiences and abstract mathematical concepts is needed. Regrouping in addition and multiplication is the first stage in a Montessori student's education where the learning objective cannot be solely learned and mastered through concrete experiences alone. This learning juncture provides an opportunity to research best practice on the use of manipulatives to support creative and expressive math learning.

The reformation of mathematics instruction in early childhood remains an underdeveloped portion of the broader conversation surrounding educational reform. Educational reformists who seek to replace a stagnant view of mathematics with one that is creative and curious must work forward from the roots of learning. Thus, research surrounding early math learning and a child's ability to develop abstract ideas and flexible problem-solving skills from the first point of abstraction in math learning is necessary. Consequently, the purpose of this action research study is to explore the impact of supplemental conceptual language on learning outcomes when performing regrouping during addition and multiplication operations in lessons that make use of Montessori manipulatives. In agreement with modern pronoun use and application, we have chosen to use gender-neutral pronouns, including they, them, and their, throughout the entirety of this paper.

Review of the Literature

Embedded and embodied cognition theory (EEC) asserts that learning is a process that is aided and enriched by concrete, physical experiences (Kiefer & Trumpp, 2012; Pouw, Van Gog, & Paas, 2014). Embedded cognition refers to learning that is aided by any tool that allows a learner to offload cognitive processing into the environment, such as taking notes or counting on fingers (Pouw et al., 2014). Embodied cognition refers to the theory that learning has a physical component and that internal cognitive processes can often be expedited by involving the body of the learner physically in some way.

Similarly, the constructivist concept of situated cognition learning is derived from an interaction between the brain, the body, and the world—EEC synthesizes principles of cognition in learning and provides a practical framework for teacher application. For example, mathematics learning that begins with a concrete, physical manipulative provides a sensorimotor experience that helps to form a schematic network that acts as an internal reference for the learner as they progress through abstractions of manipulatives and further mathematical concepts (Kiefer & Trumpp, 2012).

All human learning has a social component, a perspective originally expounded by Vygotsky called social learning theory (Schunk, 2012). As a part of the learning process, students seek sensory input from our environment; interactions with other people are included in this process. Therefore, when considering EEC outside of a strictly constructivist learning process, it is noteworthy that interactions with others are an essential component of the learning process. How students interact with their peers, teachers, and parents directly influences student learning.

With this working understanding of EEC informing our approach, three subsections of our practice as Montessori guides required further analysis. Foremost, Montessori guides are taught that their actions directly impact learning outcomes in that their physical actions will be replicated by the child receiving instruction. However, when looking at this practice critically, research indicates that instructors across pedagogies fail to reflect critically on the embedded and embodied cognitive activity happening each time a lesson with manipulatives is presented. As the embedded and embodied components of material use and procedural repetition are etched into schematic memory by the learner, their capacity to find alternative or creative strategies to the same type of problem presented in a different way reduces. Teachers' procedural modeling of math concepts is just the first step in genuinely learning that skill; the rest of the learning should be supported through conceptual language.

Research has been done on helping teachers teach for the understanding of underlying concepts, such as strong number sense and mathematical reasoning (Holm & Kajander, 2012). Number sense is a student's fundamental understanding of numbers such as quantity and magnitude, place value and proportionality. Mathematical reasoning involves the recognition of patterns and a stepwise logical approach to solving problems. The results of these studies indicate that the teacher's level of understanding and creativity in mathematics instruction impacts learner outcomes proportionally (Holm & Kajander, 2012). We can interpret this finding as follows: If the teacher is bound to procedural processes, lacking creativity and a depth of understanding, then the students will come to understand mathematics in this same rigid way.

Research on the use of manipulatives speaks to the importance of the materials in helping to create deeper math understanding in students (Pouw et al., 2014). Understanding the benefits of individual manipulatives, as well as their limitations, enables teachers to use manipulatives for the correct intended learning outcomes. Thus, working with practitioners on their understanding of how embedded and embodied cognition impacts learner outcomes may have a positive impact on both learner and teacher.

Embedded cognition theory supports the use of manipulatives as a way to lighten a student's cognitive load during the learning process (Pouw et al., 2014). Manipulatives that are too specific may hinder students' understanding by limiting their ability to transfer knowledge and develop the ability to solve problems without the aid of the manipulatives (Sloutsky et. al., 2005). The principle of manipulability of materials in EEC prompts educators to challenge and transform their use and understanding of materials to maximize the benefits these materials offer while also ensuring the acquisition of skills that are transferable and replicable.

Mathematics materials in Montessori classrooms offer a wide array of sensorimotor experiences to the student in math learning. While current research continues to validate manipulative-based instruction (Cai, 2014), EEC urges educators to look more critically at the types of manipulatives used, as well as how these tools are used. Montessori learning environments have the potential to offer rich interplay for children who receive instruction that has been analyzed using EEC theory. Our research aims to apply the principles of EEC to a social learning environment and observe the effects of this intervention on the participants.

In a 2011 study, 60% of fourth graders and 57% of eighth graders in the United States failed to demonstrate proficiency in mathematics (Carbonneau, Marley, & Selig, 2012). The statistics look even more alarming when U.S. students are compared to global populations of same-aged peers, where only 10% of fourth graders and 6% of eighth graders in the U.S. meet international standards of advanced proficiency (Carbonneau et al., 2012). National discussion on these statistics has brought some inquiry to this deficit over several years. The need to improve math education is still widely recognized in the educational research community (Brown, Lewis, Cash, Stephan, & Wang, 2016; Roselli, 2018; Thompson & Davis, 2014). It is well understood that students' mathematical learning success rests on the foundation of early learning (Cai, 2014; Thompson & Davis, 2014; Witzel, Ferguson, & Mink, 2012; Faulkner, 2009). To develop a solid base of mathematical knowledge, children must build a strong number sense and flexible math thinking skills, as well as proficiency with basic operations during their early elementary years (Faulkner, 2009; Witzel et al., 2012).

Montessori schools offer children a wealth of algorithmic math materials, which demonstrate procedural math skills in concrete terms, in conjunction with extended work cycle time. However, despite having lengthy work cycles and carefully crafted manipulatives, in practice Montessori learners and traditional learners alike demonstrate that the transition from algorithms to applied, flexible mathematics is challenging (Brown et al., 2016). Also, experienced teachers tend to continue to implement the curriculum in a controlled way, without variation in presentation or sequencing (Puchner & Taylor, 2004). Repetitious procedural work in mathematics reduces learner ability to abstract knowledge and apply learned skills (Willems & Francken, 2012). The difficulty in applying math manipulatives using collaborative, creative, discussion coupled with the lack of teacher education on alternative learning strategies, contributes to the broader deficiency in student mathematics achievement.

Since Montessori schools feature a rich array of concrete math materials, and a long-standing traditional sequence of lessons to accompany them, it has not been an active area of innovation. Dr. Montessori believed that by interacting with her didactic materials children would, "thus carry out an abstract mental operation and acquire a natural and spontaneous inclination for mental calculations" (Montessori, 1967, p. 297). Improving students' proficiency in mathematics has been addressed by defining what essential characteristics of number sense are in collaboration with educators to inform best practice (Faulkner, 2009). Within the framework of EEC, a teacher's ability to supplement the use of manipulatives impacts learner outcomes (Cai, 2013). EEC and its relationship with teacher ability, manipulatives, and number sense comprise the essential components of our research question.

Number Sense

Foundational math learning about numbers is what is referred to as number sense (Faulkner, 2009). This term has been explained by Faulkner using a graphics model that identifies the elements of number sense as to quantity/magnitude, numeration, equality, base ten, a form of a number, proportional reasoning, algebraic and geometric thinking, all arranged around the central component of language (2009). Awareness of number sense begins in young children as soon as they understand quantity and numeration. Number sense awareness develops into an ever more sophisticated understanding of new concepts, such as proportional reasoning and algebra, introduced throughout their elementary years.

Students with well-developed number sense can think flexibly about quantity and use judgment to help solve more complex mathematical problems (Sood & Mackey, 2014). These findings imply that teaching practices which emphasize procedure should be avoided, and instead, teachers should focus on fostering a deep understanding of numbers in children. From the very beginning of their math education, children should use a variety of at level tools in concert to develop their concrete and abstracted understanding of numeration, operations, and understanding of the base ten system (Parrish, 2014). As the child continues in their learning and schooling, more advanced concepts are introduced, scaffolding student learning and building off prior knowledge to further abstract their understanding of an isolated mathematical concept (Montessori, 1967).

The progression of instruction includes using language to encourage critical and creative thinking about numbers and the sub-categories found within the number sense umbrella (Faulkner, 2009). Teaching a child a mathematical skill is the first step of building math fluency, the next step is for the teacher to encourage the child to analyze and reflect with the appropriate language their understanding of the learned skill. For instance, in Figure 1, the child understood the one-to-one correspondence of the bead to a number value, knew that addition dictates the combining of amounts, and knew the number of bars and number of beads needed to reported separately. An appropriate next step for the teacher would be to have a conversation with the child that recalled prior knowledge of place value and combinations that make ten. The child is then able to analyze her work and achieve a greater understanding of the task.

Figure 1 Operations Example

The Role of the Teacher

EEC theory denotes that embodied learning experiences are a culmination of the sensorimotor experiences embedded within materials and interactions (Pouw, Van Gog, & Paas, 2014). It is impossible to separate the teacher’s influence when considering the learning outcomes of students who experience mathematical learning through direct instruction and the use of manipulatives. Hence, mathematical learning outcomes for students are greatly influenced by teacher preparedness and knowledge (Ball, 1992).

Montessori mathematics instruction for teachers and students alike is algorithmic, isolated, direct, and sequential (Montessori, 1965). Montessori math materials unfold with scaffolding ‘embedded’ in the process as the materials progress from foundational numeration to abstracted operations. However, understanding how to teach an isolated skill is distinct from crafting mathematical knowledge (Ball, Thames, & Phelps, 2008). When a learner relies on any one material to organize algorithms or mathematical processes, they lose flexibility and conceptual manipulability (Pouw et al., 2014).

Teaching for conceptual understanding is counter to the procedural process most teachers, including Montessori teachers, were trained on themselves, making it difficult to re-educate facilitators of mathematical manipulatives (Faulkner, 2009). Reframing how Montessori educators perceive the mathematics manipulatives by defining the materials’ limitations, and encouraging creative, adaptable math conversations may have a positive impact on learner outcomes (Yoon, Duncan, Lee, Scarloss, & Shapley, 2007). Success in teaching with manipulatives may require the teacher to explicitly connect the material and procedure to the abstract idea (Pouw et al., 2014). The manipulatives themselves offer embedded information that the learner can embody with appropriate instruction and support, but the manipulatives cannot connect them to the abstract idea.

A conversation between student and teacher about what is being learned is what connects the learner to abstract mathematical ideas. For example, Ball (1992) recounts an impromptu learning exercise chosen by her students when learning about odd and even numbers. Initially, Ball used manipulatives to teach the ‘right’ concept of odd and even (1992). One student challenged the lesson by stating that some numbers could be odd, *and* even, the students then discussed different ideas of what was odd and even (Ball, 1992). The reasoning of students who offered different, non-traditional groupings was logically sound, and thus Ball realized that manipulative instruction only took learners so far. It was the conversation that followed manipulative use that brought the learners to *know* odd and even and their applications in practical mathematics.

Teaching conversational mathematics learning strategies can be the link between EEC and higher academic performance (Parrish, 2014); and so, when supplementing what manipulatives lack in practice with proven learning strategies, it offers the learner more profound learning experiences. As a result, best practice for educators should be to invest in learning about how EEC in conjunction with supplemental conversational strategies can increase learner outcomes.

Manipulatives

Researchers include the use of manipulatives in teaching mathematics as part of ‘best practice,’ particularly as it relates to foundational math learning (Carbonneau et al., 2012). Manipulatives are useful as they encourage the learner to be physically involved in the learning process. Manipulatives present concepts of quantity in concrete terms, allowing young learners, students ages four to nine, to offload some of the cognitive burdens to the environment, isolating their effort to the idea presented within the material (Pouw et al., 2014).

Involving children ages four to nine physically in learning math concepts may aid them in their construction of meaning since younger learners are more dependent on concrete processes (Montessori, 1995). Self-performed tasks enhance subsequent information retrieval, so children who learn with concrete manipulatives may have better recall of information learned (Carbonneau et al., 2012).

Conversely, critics (Ball, 1992; Ball, Thames, & Phelps, 2008; Pouw et al., 2014; Torbeyns & Verschaffel, 2015) highlight the risk of expecting a child's interaction with given concrete material to "magically" impart mathematical knowledge. Further, Ball (1992) contends, "understanding does not travel through the fingertips and up the arm" (p. 47). Research substantiates this claim, as the success of the use of manipulatives declines when they are used without direct instruction (Carbonneau et al., 2012). A moderate perspective at the impasse of pure constructivism and direct instruction would be this: concrete manipulatives are a tool for teaching, not the sole vehicle of conceptual learning.

Additionally, there are practical concerns regarding the reliability of modeling basic mathematical concepts through manipulatives alone (Ball, 2012). Children are naturally creative and reflexive thinkers; educators often find themselves in precarious learning situations where a child demonstrates rational, logical thought, which supports a mistaken idea or answer. Perhaps the most apparent shortcoming of manipulatives in early math learning is that the materials themselves cannot encourage critical thinking (Belenky & Schalk, 2014). For example, a child adding seventeen and six using a concrete bead material would take out a ten bar and seven individual beads, as well as six individual beads. Following a procedure that called for the adding of the beads, the child might write 113 as the answer (see Figure 1).

The rigidity of using manipulatives in math instruction seldom includes an extensive conversation about different strategies that lead to correct or incorrect answers. The materials must be used in conjunction with lessons that encourage reflective conversation and critical thinking that connects to prior knowledge.

As a result, concern has also been raised that learning mathematical concepts with manipulatives prevents learners from developing their conception of how to solve a given problem (Pouw et al., 2014). This idea feels counterintuitive to what most educators learn about constructivist learning practices and manipulatives. Within a constructivist framework, manipulatives that isolate a specific learning concept provide the learner *experience* of that concept, enabling them to internalize mathematical concepts and processes to prepare for further abstraction (Montessori, 1995; Schunk, 2012). However, research on this accepted norm has proven that often, manipulatives fall short of this learning expectation (Pouw et al., 2014). Learners who rely on specific manipulatives to retrieve information about mathematical processes cannot apply the learned ‘concepts’ to different materials or questions (Belenky & Schalk, 2014).

Manipulatives are a staple of mathematics learning that should be included in a variety of means and modalities in early mathematics education. The literature suggests that although the efficacy of manipulatives as a learning tool is not in question, their practical implications and versatility are in question. The whole idea of number sense provides insight to educators as to how to improve the use of manipulatives and promote flexible, critical thinking.

The purpose of this research is to investigate how the use of math manipulatives informed by EEC theory and supplemented with conceptual mathematical language, situated within the overarching model of number sense, can interact to improve student learning outcomes. Through the incorporation of conceptual mathematical language, we predict that students will gain a more in-depth insight into mathematical processes, increasing learner outcomes.

Methodology

This study used an experimental design. Qualitative measurements including observation, informal interviews, focus groups, and checklists were used to assess the relationship between conceptual language and student outcomes when performing regrouping in addition and multiplication. Pre and post rubrics (Appendix A) were used to interview student participants. The interviews included questions about the process of regrouping in addition and multiplication, as well as inquiries about student strategies, articulation, and level of understanding.

The population for this action research study was Montessori students enrolled in the primary (3-6-years-old) and lower elementary (6-9-years-old) divisions of two different private Montessori schools (N=124) in the United States. The sample size was 29 total, consisting of six second year primary students, nine third year primary students, six first grade students, six second grade students, and three third grade students. The sample featured 14 boys and 16 girls. Subjects were randomly selected and were representative of the population of students within the primary and lower elementary programs.

Pre and post assessments were used in an interview schedule that centered on inquiry-based conversation designed to gain insight into student understanding about the process of regrouping during addition and multiplication. Specifically, the researchers determined to what extent the student understands place value, can use mathematical language in context, can offer alternative strategies, can articulate the process of regrouping, and can decompose the sum and identify component parts of the problem.

Focus groups were held with the lead classroom teachers before and after the intervention to determine their perceptions of student understanding on exchanging in addition and multiplication. The focus groups were guided by previously outlined questions pertaining to mathematics and mathematic material use (Appendix B). Teachers were asked questions about their perceptions of material use, how they think their students are progressing in exchanging through different operations, and what additional observations they have collected throughout this process.

Observations were taken throughout the study period using previously outlined prompts (Appendix C & D). The prompts were used by classroom teachers and researchers, to capture any evolution or spontaneity in students' use of the mathematics materials. The prompts are short, allowing the observer to respond concisely. The prompts look at math skills such as student audible self-talk, strategies used, type of material used, etc.

Lastly, researchers utilized a checklist to track student performance during one on one work throughout the treatment process (Appendix E). The checklist allowed the researchers to quickly assess whether the student recognized place value, could offer alternative strategies, could use mathematical language in context, verbally articulate the process of exchanging, and decompose the sum and identify component parts.

Students were assessed individually in the first week of the study using a pre-intervention rubric. During the first week of the intervention process, researchers met with participating lead teachers in a focus group. The researchers used the focus group question form to guide the conversation to include observations and opinions about math material use, student learning and progress in the process of exchanging, and how teachers perceive mathematical material use in their classrooms.

Treatment took the form of lessons given weekly by the researchers to students in the sample group on dynamic addition, and when appropriate multiplication. Dynamic addition and multiplication lessons include the use of the Golden Beads (Appendix F), which is a Montessori math material. The golden beads consist of individual unit beads, bars which have ten unit beads, squares which have one hundred unit beads, and cubes which have one thousand unit beads. Often, the golden bead materials are kept in a special space in the math area of a classroom called 'the bank'. The golden beads are the most concrete of the math manipulatives found in Montessori classrooms, and all four operations can be introduced and practiced using this material. 'Dynamic' is the word used by Montessori practitioners to describe the process of exchanging or regrouping within operations. The golden beads can be combined to demonstrate the process of regrouping in a tangible way.

Students who were already familiar with dynamic addition and multiplication were asked to use the materials in their work and to explain to the researcher what they were doing as they completed one to three problems. Students who were new to dynamic addition were given appropriate introductory lessons. Each lesson with a student included one to three problems, and researchers engaged the students in conversation at every lesson by introducing or reviewing conceptual language. Specifically, conversation included the terminology of exchanging, which is the term used in Montessori classrooms to describe the action of taking ten of one denomination and making an even exchange at the golden bead bank for the larger place value representative of the quantity. For example, this terminology is used when trading ten ten-bars for a hundred square, and ten hundred-squares for a thousand cube.

Students who felt they did not need to use materials to complete the work were allowed do one of the problems in the problem set using paper, pencil, and mental math. Furthermore, the student was asked to describe how they solved the problem as they worked through it. Throughout the process of working with students individually, researchers marked down student performance on a checklist at each lesson. Additionally, during the intervention period researchers conducted a once weekly observation using the previously developed observation form (Appendix D). Furthermore, teacher participants also completed once weekly observation forms throughout the intervention process. Individual student assessments using a post intervention rubric were given in the last week of the study. Following the post intervention assessment, researchers met with participating lead teachers in a final focus group to discuss observations, utilizing the focus group questions form to guide the conversation.

Researchers analyzed pre and post treatment responses on the checklist and pre and post intervention rubric assessments to assess student skill development in the process of exchanging in addition and multiplication. The data these tools collect revealed average numerical improvement in student understanding within identified components of mathematical competency, as well as helping researchers identify important patterns across the intervention. Observational data and focus group responses added to the broader picture of mathematical learning in the classroom as well as insights the intervention process revealed.

Analysis of the Data

The purpose of this study was to learn about the potential impact of using conceptual language during addition and multiplication operations performed by Montessori students ages 4-9 years old. The research design incorporated both quantitative and qualitative data tools such as observational notes, focus groups, checklists, and a pre and post assessment rubric which measured student ability in areas such as recognition of place value, articulation of the process of exchanging, and the ability to decompose sums and identify component parts.

The subjects for this study were Montessori students ages three-to-nine-years-old enrolled in two separate private schools in the United States. A total of six classrooms were surveyed beginning in September 2019. From the total number of students (N=124), five students from each classroom in each school were randomly selected for participation. Twenty-nine students total participated in the study, sixteen were female and thirteen were male (see Figure 2).

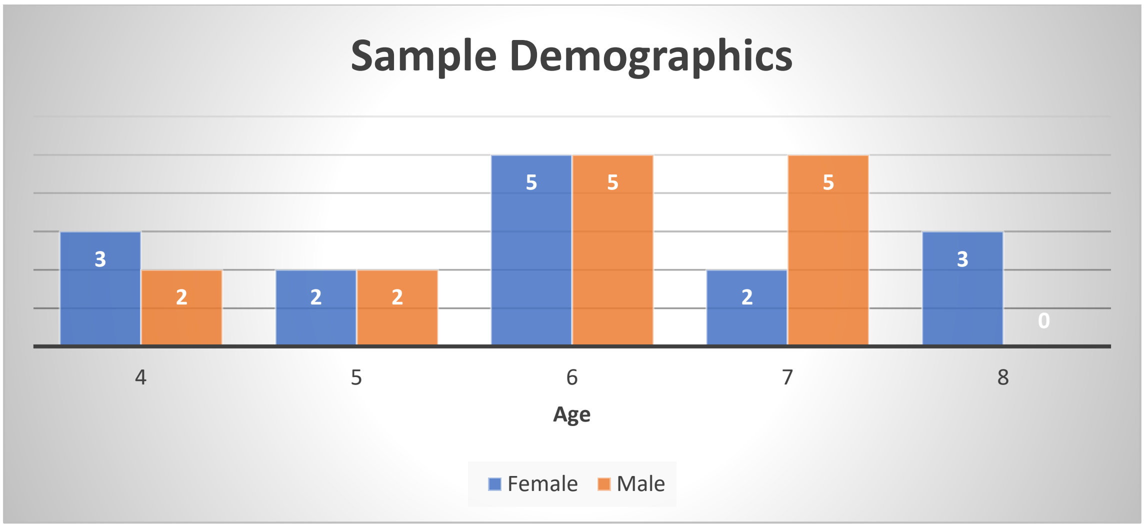

Figure 2 Sample Demographics

Articulating Exchanging

The first research question that this study addressed was the learner's ability to articulate the process of exchanging. To answer this question, the researchers asked probative questions such as, "What does 'exchanging' mean?", and, "How would you describe exchanging?", and recorded student responses. The researchers used a pre and post assessment rubric and a checklist to measure the degree of ability and frequency with which the children could define exchanging. Student responses were entered in Excel and analyzed to determine if the intervention strategies had any impact on student response.

Table 1

Articulating Exchanging:	**Pre-Assessment**	**Post-Assessment**	**Change**
Articulates Steps (ages 4-6)	1.14	3.21	+2.07
Articulates Steps (ages 6-9)	2.53	3.73	+1.20
Uses Appropriate Language (ages 4-6)	1.50	3.50	+2.00
Uses Appropriate Language (ages 6-9)	2.20	4.07	+1.87

Pre and post-assessment rubric scores on a scale of 1-5.

The data indicates that children ages 4-6-years-old had a larger improvement than their 6-9-year-old peers in articulating exchanging and using appropriate language when describing operational processes. Of the two variables listed in Table 1, students improved by roughly two points within the rubric scoring system. Both the trend of a two-point increase in outcome and younger students improving slightly more than their older peers is consistent across data points.

For example, data points which complement the articulation of exchanging include the ability to identify four place values and arriving at the correct answer after explaining verbally reinforce the data trend found in Table 1. On average across the four data points identifying knowledge of place value, participants ages four to six years old improved by 2.28 points. Comparatively, participants ages six to nine had a 0.17-point average improvement across the same four data points. Participants ages four to six had a 1.93-point increase in arriving at the correct answer when explaining the operation verbally while participants ages six to nine had a 0.67-point increase.

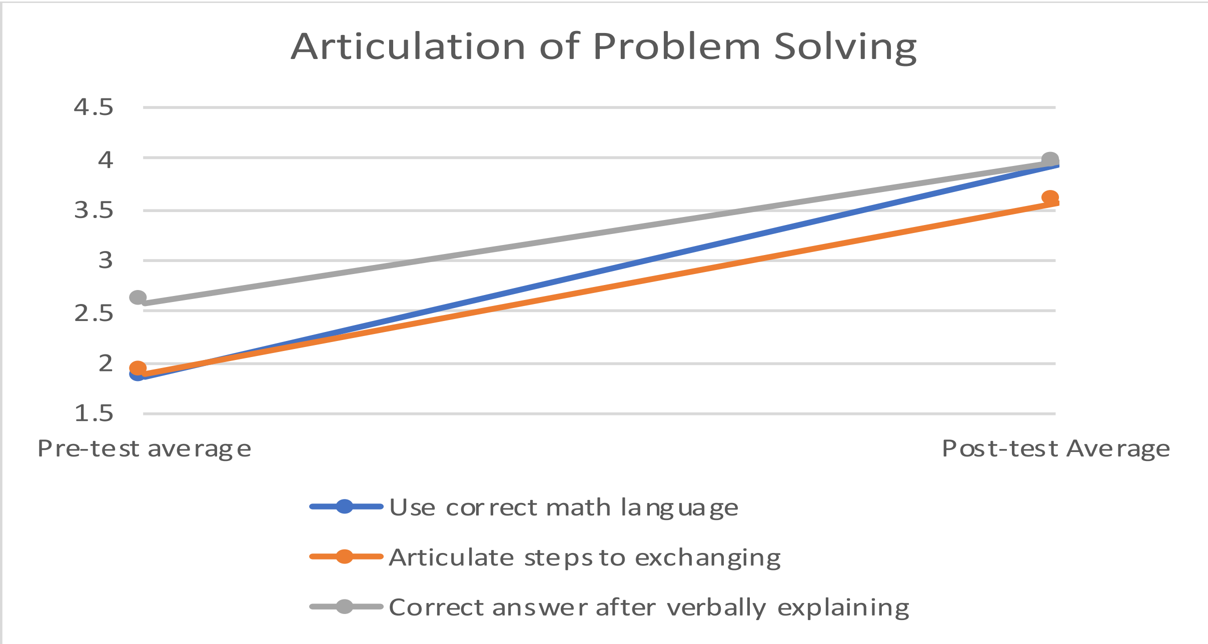

Figure 3 Average scores on 5-point rubric pre- and post-assessment across 4-9-year-old participants

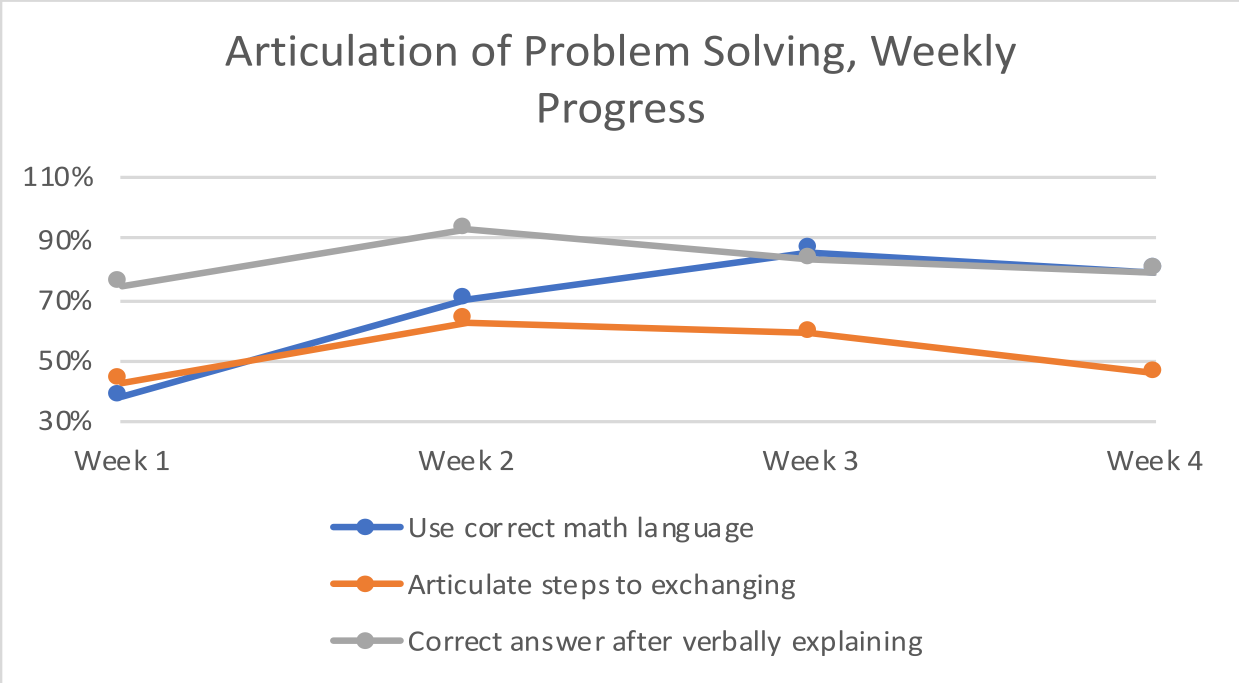

Figure 4 Percentage of students able to demonstrate skill

Taken as a whole, students showed improvement in the skill of articulation of exchange from the pre- to the post-assessment, although there was some variability in week to week progress (see Figure 3 and Figure 4).

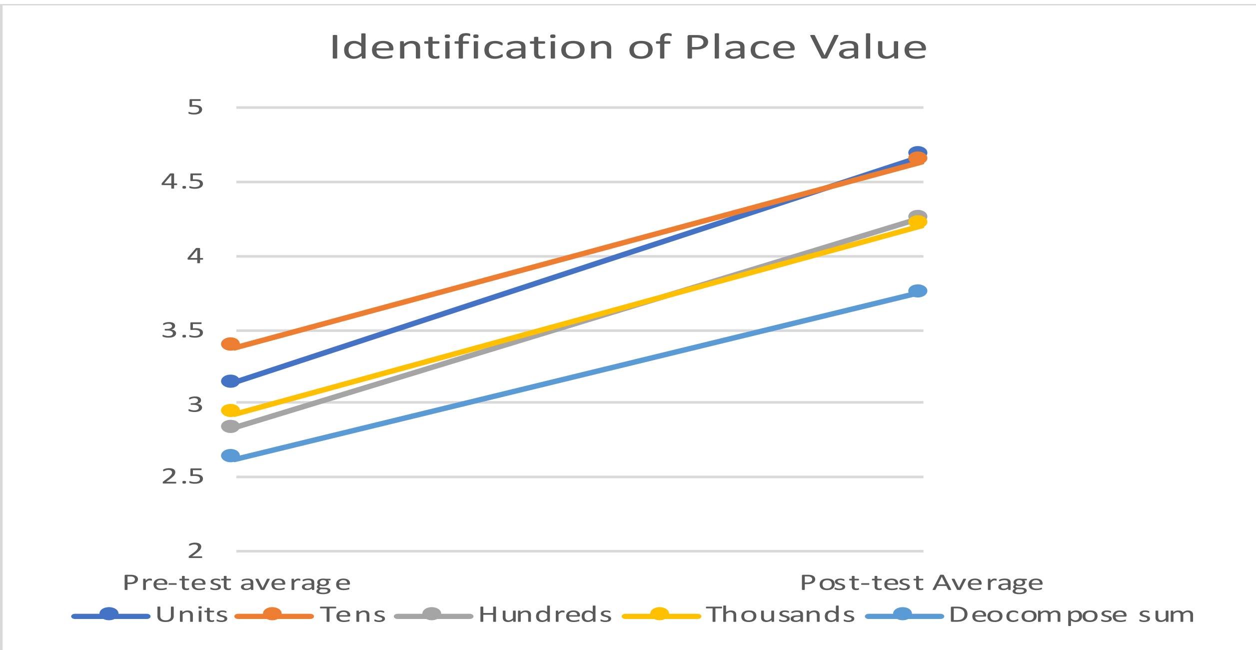

Figure 5 Average scores on 5-point rubric pre- and post-assessment across 4-9-year-old participants

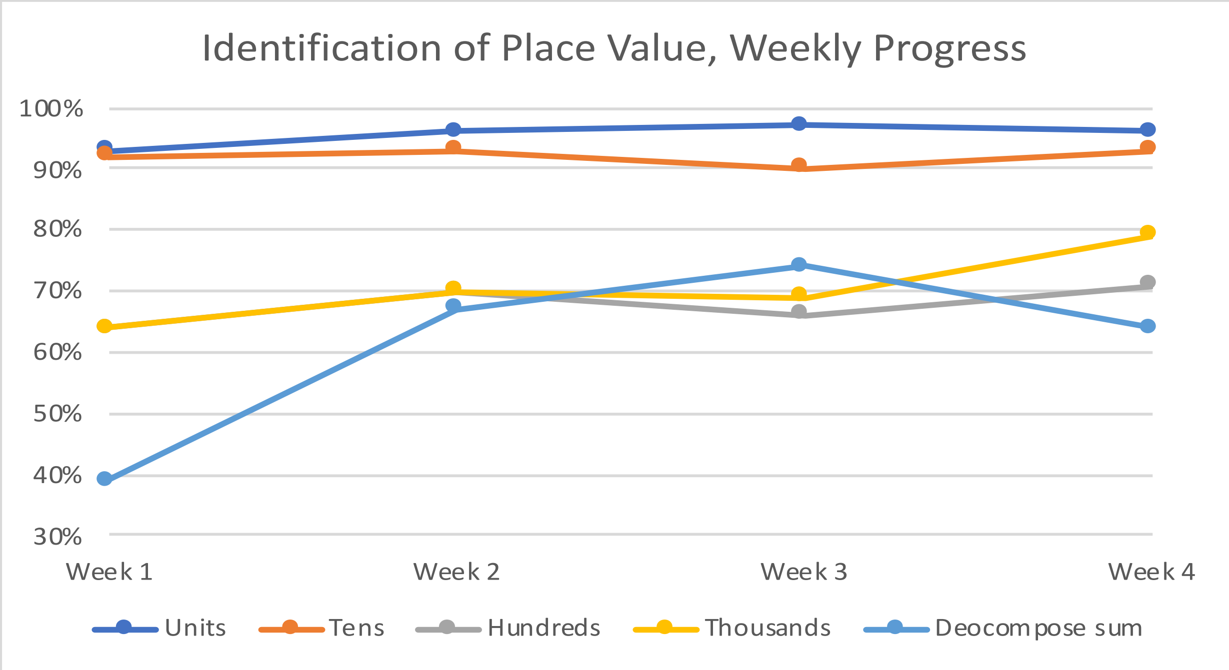

Figure 6 Percentage of students able to demonstrate skill

As a group, students showed clear improvement in the ability to identify place value which is demonstrated in Figure 5. The weekly progress revealed in Figure 6 indicates that some learning variables showed less improvement because students were already proficient in the skill.

Alternative Strategies

The study also sought to determine if incorporating conceptual language into addition and multiplication lessons would have an impact on the students' ability to offer alternative strategies during these operations. To answer this question, researchers asked the children directly, "Can you think of a different way to solve this problem?", "Is there a way you feel more comfortable using these materials?". The children's responses were recorded on a pre and post assessment rubric (Appendix A) which measured ability, and on a checklist, which recorded frequency in offering alternative strategies during operations work. Student responses were entered in Excel and compared to determine if the intervention had an effect.

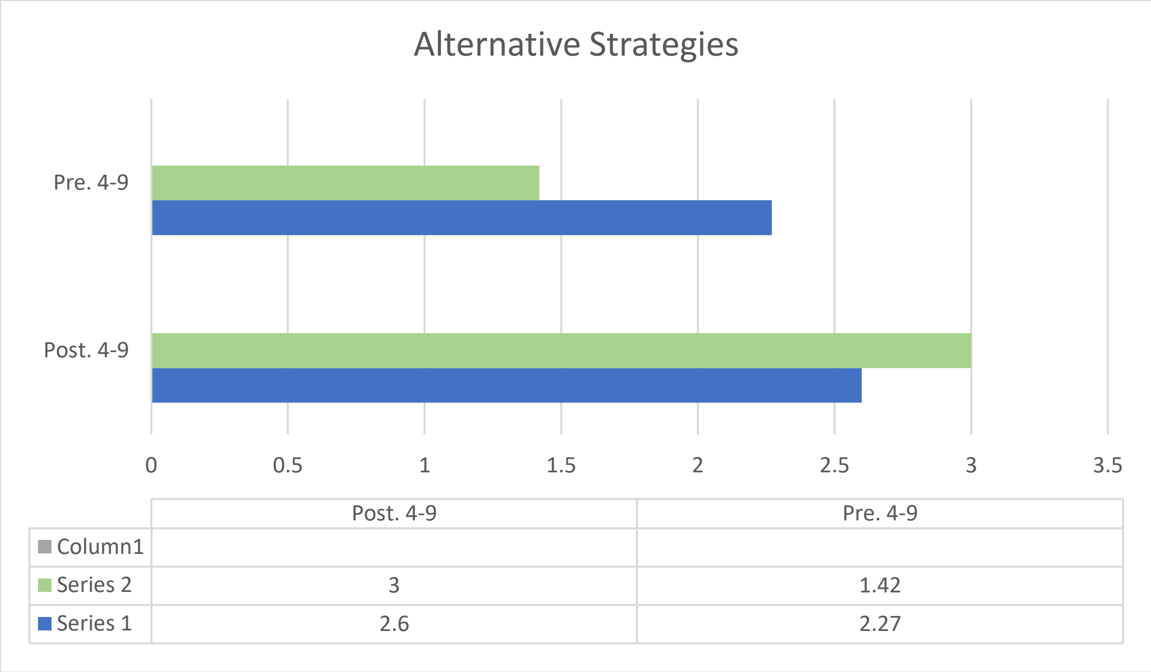

Figure 7 Students' ability to offer an alternative strategy to solving an addition problem

Figure 7 indicates that series one, comprised of 6-9-year-old participants, had a small amount of growth in being able to offer alternative strategies, totaling 0.67 points. Series two, comprised of 4-6-year-old participants, also showed improvement by 1.58 points. The learning variable of offering alternative strategies showed the least amount of change across data points.

Perspectives on Abstraction

Lastly, this study aimed to gain insight into how Montessori teachers perceive the development of abstract mathematical ideas. The researchers engaged in two focus groups, one before the intervention process and one after. Additionally, teacher participants were given the option to complete a weekly observation form. Responses from the observation forms and the focus group meetings were analyzed by the researchers, and prominent themes were noted. Early childhood (3-6-years-old) and lower elementary (6-9-years-old) teachers identified three areas of mathematical learning that could be explored further when studying operations with the golden beads: persistence, understanding, and desired outcomes.

The first identified area, persistence, was observed differently between the two age groups. In the 3-6 classrooms, teachers felt that children lacked either the ability or the will to persist through the long procedure required when working with the golden beads. At the beginning of the study, 3-6 teachers identified this as a learning gap, something that could be managed through repetition and exposure to procedure. At the end of the study, they shifted this perspective to one of questioning whether the children were responsible for this learning gap or if the materials simply lacked inspiring or motivating qualities.

Teachers in the 6-9 learning environments felt that the children did not persist through the problem by deferring to ‘easier’ strategies. For example, one of the children may have been able to arrive at the correct answer using strictly abstracted paper to pencil math but could not arrive at the correct answer using the golden beads. Teachers in the 6-9 classrooms concluded that revisiting the golden beads more often during operations may be a solution to the lack of persistence in combining materials and alternative strategies.

The second identified theme, understanding, reflects shared conclusions and perspectives from all teacher participants in the study. Building from observations surrounding persistence, teachers also noticed that there is an apparent chasm in understanding involving quantity, place value, and procedure. For instance, children might be able to read a large number without being able to produce the corresponding quantity. Children might perform an exchange without understanding that the ten tens they just put away did not disappear, but rather were moved into the larger place value and are now reflected as one hundred. Children might be able to arrive at the correct answer doing ‘column math’, solving a problem by place value, without understanding that they are adding together two composed numbers to find a sum. These types of observations were plentiful, leading the teachers to conclude that understanding during operational work involving exchanging requires further examination and testing.

Lastly, teachers identified desired outcomes as a theme during observation and reflection. By desired outcomes teacher participants meant what do we hope the children will be able to do once they have mastered this lesson. Early childhood teachers (3-6) all shared the sentiment that because they were unsure of the golden beads' value in learning operations at this early learning stage, they needed to re-evaluate their desired learning outcomes for this material. They suggested different testing that could be done independently in their classrooms post-study to assess the effectiveness of the material. Additionally, they recognized that if they wanted children to master long procedures, there were other materials in the classroom which meet that developmental goal. For early childhood teachers (3-6), the conversation evolved into a philosophical conversation about the intrinsic value of sensorial learning.

The lower elementary participants (6-9) assessed their desired learning outcomes by incorporating their observations from the identified themes of persistence and understanding. In their observations of the children, they found that using the golden beads to remind the children of quantity grounded the operational process, supporting a deeper understanding of the math being performed. The lower elementary participants (6-9) pondered how their students' developmental phase influenced their ability to abstract mathematical procedures, and what that meant when applying concrete materials to this process.

Discussion

The purpose of this study was to explore whether using supplemental conceptual language during math lessons on exchanging in addition and multiplication would have an impact on learner outcomes. The researchers in the study used a combination of quantitative and qualitative data collection tools including a pre and post assessment rubric, a checklist, observation forms, and focus group to measure the intervention process. The data collected throughout the study indicates a positive improvement in learner outcomes with 4-6-year-old participants demonstrating a two-point increase across learning variables, and 6-9-year-old participants showing a more modest improvement of 0.6 across learning variables. Children ages 6-9 started the study with higher basal knowledge than their younger counterparts, which contributed to their more modest point changes across learning variables.

Upon further analysis of the data within the context of EEC theory and additional mathematical education literature, the findings support EEC theory and mathematical literature critiquing the isolated use of concrete materials. EEC theory asserts that the embedded and embodied components of a material do not lend themselves directly to abstracted or creative learning (Pouw et al., 2014). This assertion is supported by the research within identified learning variables such as offering alternative strategies, (see Figure 5 and Figure 6). Students had the least amount of growth in this area overall, offering few deviations from standard lesson procedure. Additionally, an anecdote from an observation of 4-6-year-olds notes, “The child could not complete the procedure because she was distracted by the difference in work mat color.” Typically place values are identified by color coded mats when working with the golden beads, when the golden beads were placed on a white work mat, the student lost all ability to move forward with procedure.” How strictly the children are taught to use the materials to arrive at the correct answer appears to proportionally reduce their ability to apply learned strategies and procedures to different situations.

Further, through the assessment of both numerical and observational data, the research indicates that despite an improvement in learner outcomes across the intervention the children still struggled with the permanence of quantities in isolation and during exchanging. For example, an older child may be able to read a large number such as one thousand two hundred eighty-four, but when asked how much that is, they struggle to verbally decompose that number and overlook deferring to the concrete materials to demonstrate the quantity. It is as if the quantity, though accessible and real on the paper, does not exist as a substantive, real quantity in their mind.

Another example, a child (4-9) might exchange ten tens for one hundred when performing an exchange, the child may say, “I got rid of the tens.” This child holds the understanding that the tens they just exchanged are no longer part of the operation, despite having the ten tens in hand in the form of one hundred. In both older and younger children, permanence of quantity is an important theme extracted from the data. This is congruent with EEC theory which asserts when repetitious use of materials is present without supplemental language or strategies, the learner becomes confined to the embodied process tethered to the materials (Pouw et al., 2014). The data bears witness to this process as we explore the theme of permanence in exchanging during addition and multiplication operations.

Based on the findings of this study, three distinct conclusions were drawn. Firstly, procedurally based learning can hinder creative and reflexive thinking. For both age groups, proficiency in procedure, and the absence of direct instruction on alternative problem-solving strategies, led to students explaining exchanging in a singular and isolated fashion. Additionally, students who were able to identify place value and perform the procedure of exchanging lacked a depth of understanding when questioned about their process. This means that not only did ‘proficient’ children offer only one strategy, they could not articulate what that strategy meant in relationship to the operation before them.

Torbeyns and Verschaffel found similar conclusions in their study assessing the algorithmic choices of fourth graders, noting that standard algorithms were applied more frequently than alternative strategies (2015). Furthermore, the study indicated that children who did employ alternative strategies chose to do so based on individual exposure and mastery of procedure (Torbeyns & Verschaffel, 2015). This is consistent with studies inquiring about manipulative use and efficacy, whose findings indicate that manipulatives can hinder creative and reflexive thinking if not introduced in multiple ways (Pouw et. al., 2014; Thompson & Davis, 2014; Carbonneau et. al, 2012).

The second conclusion drawn from the study was that the golden bead material may have alternative uses based on age groups. Teacher participants in 3-6-year-old classrooms felt that the golden bead materials may need further exploration to determine how their application would be most beneficial for learner outcomes. They concluded that they were not convinced of their efficacy at this early stage of mathematical development. In contrast, 6-9-year-old teacher participants found that returning to the golden beads, a material that is less used by older students, was beneficial to helping them explain the procedure they had become proficient in solving using just paper and pencil. Additionally, they found that the golden beads were a useful supporting material when teaching operations with further abstracted Montessori manipulatives.

The findings of teacher participants from the six-to-nine-year-old classrooms are consistent with findings of longitudinal studies conducted with older children as the sample group (Torbeyns & Verschaffel, 2015; Fuchs et. al, 2010). Variation, repetition, and the use of concrete manipulatives does appear to further primary aged student learning across curriculums and interventions. However, no studies citied within this work specifically address children in pre-primary ages classrooms. Thus, the findings contributed by three-to-six-year-old teacher participants might be explored through further testing to improve this less explored aspect of mathematics education.

Lastly, the use of supplemental conceptual language is critical to developing appropriate use of nomenclature and creative thinking. Without supporting embedded and embodied procedural steps with language that helps the child construct understanding of the abstract concepts being taught, the child will lack the internal framework with which to apply alternative strategies. Mathematics studies which focused on teacher efficacy and ability underscore the importance of teacher preparedness and our effect on learner outcomes (Thompson & Davis, 2012; Cai, 2013; Puchner & Taylor, 2006). Children who had the least exposure to procedures involving operations, demonstrated the most growth throughout the intervention. The language they learned alongside the procedural processes furthered their learning instead of dismantling it. Their peers who understood procedure struggled to supplement their learning with conceptual language, the procedure was so ingrained into the outcome of the problem, it was as if the problem has lost meaning outside the context of the procedure.

Recommendations

Based on the findings and conclusions of this study, four recommendations were made. Primarily, even the most comprehensive set of manipulative learning aids, such as the golden beads found in Montessori classrooms, have limitations of which teachers should be aware. The materials cannot instruct the children in isolation, conversations to clarify student understanding and contextualize math learning objectives are an important step in the learning process. These conversations should include proper math vocabulary and include questions such as, “Can you tell me about how you solved this problem?”. Specifically, older students who turn in written work for review should be challenged to articulate their processes and strategies, as well as occasionally demonstrate their abstracted work using concrete materials.

Additionally, presentations of procedural lessons in mathematics should be varied. It is common practice in the Montessori method that lessons have a ritualized quality, which is helpful in teaching children procedures and independence. However, it is important to introduce novelty into math lessons to promote flexibility in student thinking. This novelty deepens student understanding of the universal applicability of math concepts. For example, occasionally presenting a problem horizontally rather than stacked vertically can help a child to recognize and identify that two quantities are being combined in both orientations, and that there are alternative ways to approach and solve such problems.

Furthermore, teachers should promote student understanding of the permanence of quantity in exchanging. To address this, teachers might impose a greater emphasis on understanding place value deeply. The study observed that specifically older students, who may be proficient in the procedure of solving exchanges in addition and multiplication, may be unable to articulate that process due to a shallow understanding of place value. A proposed solution to this area of inquiry could be a conversation between early childhood (3-6) and lower elementary (6-9) teachers considering the potential impact of early childhood (3-6) classrooms employing a strictly sensorial training in the golden beads. Leaving procedural and abstracted work to the lower elementary (6-9) teachers and students.

Additional interventions might include grouping children specifically for the purpose of discussing math and brainstorming alternative ways to solve a math problem. One 6-9-year-old teacher participant in the study had observed the difficulty some of her students had in explaining their work during the research process and proactively started to group children to work on math. She reported that there was a subsequent increase in math related discussion among the children. This would be an interesting area of further study as it holds the potential to be an avenue to help children articulate math learning and promote flexible thinking by sharing ideas for alternative thinking about solving problems.

This small study was employed under a limited intervention period. A longer intervention may have amplified the positive trends that were seen in our results. Our results did provide insight into the challenges and potential deficits in math learning seen in our classrooms, and potentially all similarly operating Montessori classrooms. Several alternative ideas have come to mind during the study.

The model of this study could be used and replicated to assess other areas of operational work with the golden beads, or isolate initial lessons in the golden beads to measure their efficiency. At the conclusion of this study, we have more questions than answers, and we implore the broader Montessori and educational community to discuss not only the usefulness of the golden beads and other concrete manipulatives, but to reflect critically on the purpose and outcomes associated with all systematic components of the Montessori mathematical curriculum. Let us defer to the children in determining which areas of the classroom are complete or are worth further inquiry, observing honestly the areas of growth that we as educators are responsible for evaluating in order to create an environment with true academic integrity and respect for the development of the child.

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Appendix A

Class Name Here					
Pre-Test: _________ Post-Test___________					
1 = No literacy	2 = Developing literacy	3 = Proficiency	4 = Mastery	5 = Exceeds Expectations	
STUDENT NAME	A	B	C	D	E
TASK	TASK SCORE (1 TO 5 SCALE)				
Student demonstrates understanding of unit place value verbally					
Student demonstrates understanding of tens place value verbally					
Student demonstrates understanding of hundreds place value verbally					
Student demonstrates understanding of thousands place value verbally					
Student verbally uses language including add, carry, and exchange when describing procedure					
Student is able to articulate the steps to exchanging without concrete materials to guide the procedural process					
Student is able to offer alternative strategies to the lesson procedure					
Student can verbally decompose sum and identify component parts					
Student arrives at correct answer using the manipulatives before explaining verbally					
Student arrives at correct answer after explaining verbally					

Appendix B

Focus Group Questions

1) What have you noticed about the relationship between the math materials and abstracted pencil and paper calculations?

2) Do you feel the children draw explicit connections between the materials and the abstract concepts?

3) How well do you think the children understand the process of exchanging?

3a) Are they able to explain what they are thinking as they solve problems?

4) Please comment specifically on your perception of the current level of understanding of exchanging for each individual child you work with who will participate in our study.

5) Do the children speak with each other about strategies? If so, what type of things do you hear them saying to each other?

Appendix C

Reflection Form **Date ______________**

How many children worked on target areas of addition or multiplication with regrouping this week?	
How many direct lessons did you give this week in the target area of addition or multiplication with regrouping?	

Reflection on a lesson done this week in target area:

Form of work: ___ Materials only ___ Abstraction only ___ Materials and Abstraction

Which material(s) did the student use?	
Did the student engage in audible self-talk?	
Did the student engage in conversation about the procedure on their own? Or did the child require prompting in order to articulate the process of exchanging?	
Did the student use any visible strategies?	
Did the student arrive at the correct answer?	

Any additional observations?

Appendix D

Field Notes **Date _____________**

How many children are working on math?	
How many children are working on target areas of addition or multiplication with regrouping?	

Observation: Child's Name _________ Age _________

Form of work: ___ Materials only ___ Abstraction only ___ Materials and Abstraction

Which material(s) is the student using?	
Is the student engaging in audible self-talk?	
Is the student engaging in math-related discussion with peers?	
Is the student using any visible strategies?	
Did the student arrive at the correct answer?	

Any additional observations?

Appendix E

Intervention checklist

Date: ___________ Time: ___________

Student Name and Age: ___________________ Lesson: ___________________

Task	Yes	No
Student demonstrates understanding of unit place value		
Student demonstrates understanding of tens place value		
Student demonstrates understanding of hundreds place value		
Student demonstrates understanding of thousands place value		
Student uses language including add, carry, and exchange when describing procedure		
Student is able to articulate the steps to exchanging without concrete materials to guide the procedural process		
Student is able to offer alternative strategies		
Student can decompose sum and identify component parts		
Student arrives at correct answer before explaining verbally		
Student arrives at correct answer after explaining verbally		

Appendix F

The Golden Beads