

Chaos Theory

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Chaos Theory

Tatiana Olson

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Abstract

Chaos theory is the study of change over time, specifically of highly volatile, seemingly random situations that are highly sensitive to changes in initial conditions. Fractals are self-similar objects with non integer dimensions. In this paper, I will explore the history of chaos theory, fractals, and the relationship between them.

A dynamical system is one that models the change over time, each value is dependent on the previous. That is $x_{t+1} = f(x_t)$. Sometimes these systems are orderly, symmetric, and predictable.

These systems can also be chaotic. There are three characteristics of a chaotic system. First, it has a dense collection of points. Second, it is highly dependent on initial conditions. Third, two points that start out close together will end up far apart.

Edward Lorenz discovered this while modeling weather in 1961. He used the various weather conditions and stored the results in memory to six decimal places. He then printed out the results and started over from the beginning. When he printed out the results a second time to examine the results a point of being unrecognizable. Two points that were only 0.000001 units apart (0.506127 vs 0.506128) diverged.



Figure 1: [2]

Prior to this, it was assumed that small changes in initial values would have minimal effects on the final value. For chaotic equations this is untrue; small differences quickly build up on each other. Unfortunately, this shows that weather is unpredictable more than a few days in advance no matter how good the data is [3]. Edward Lorenz next looked at convection, the phenomenon of hot air rising and cool air falling. Using a simplified version of equations based off of work by B. Saltzman, he came up with a system of three differential equations

$$dx/dt = -10x + 10y \tag{1}$$

$$dy/dt = 28x - y - xz \tag{2}$$

$$dz/dt = 8/3z + xy \tag{3}$$

in which x , y , and z were key variables and t was time. Ignoring the xz and xy terms, allowed Lorenze to look at the steady states of the system, characterized by x , y , and z remaining constant. In this system, he found that one indicated no convection and the other two indicated steady convection. Linear stability analysis allows analysis of the stability of the system near the steady state points. The system must be solved to look at what happens away from the steady states. Lorenz used a computer to solve and map the solutions. The result in the x y plane is a two lobed figure which resembles a butterfly. This is not related to the term butterfly effect however. The term butterfly effect is in reference to the second characteristic of chaotic equations, that they are highly sensitive to changes to initial conditions. Thus, something as simple as a butterfly flapping its wings can affect the weather on the other side of the planet. Lorenz published his findings on chaos in

the Journal of the Atmo
that his work went unkn

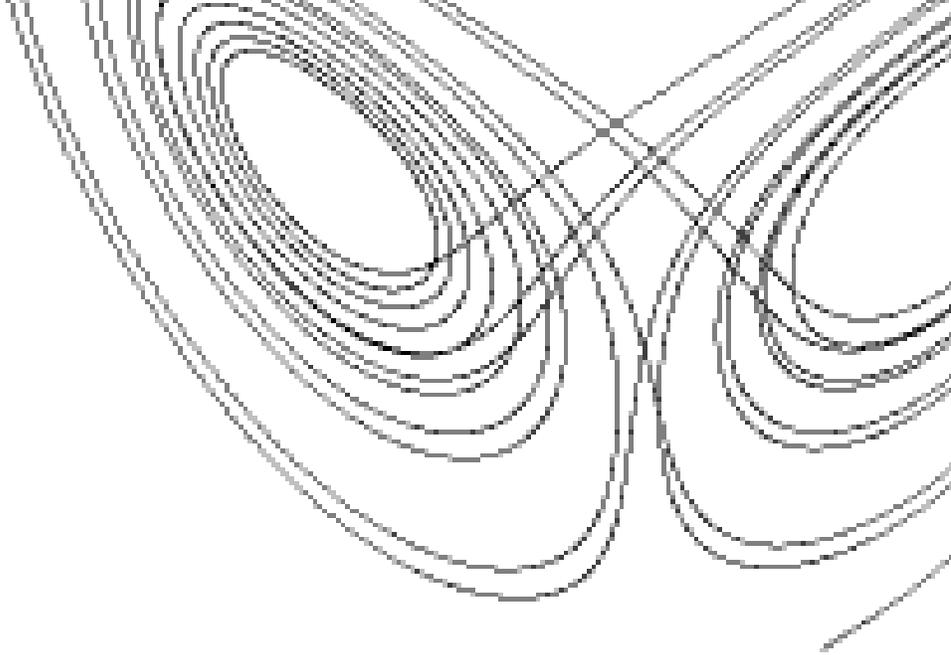


Figure 2: Map in the xy plane [2]

An example of a simple chaotic system is $f(x) = ax(1 - x)$. For $a < 1$, the equation converges to zero. When $1 < a < 2$, the equation has a fixed point, a point that maps to itself, besides zero. That is, for some x_i , $x_i = ax_i(1 - x_i)$. The following interval, $2 < a < 3$, changes the behavior around the fixed point to the cobweb, where the values spiral around the fixed point. When a is larger than three, chaotic behavior appears [5].

An example presented by Stewart is $2x^2 - 1$ which takes initial values between zero and one. To show how quickly these solutions to these equations will diverge with two slightly different starting points I used Excel to compute it two hundred times each for both initial values. I used Excel's stored value of pi (fifteen digits) and subtracted 3 for my first value and 0.1416 for my second initial value. The chart below, to fifty iterations, shows how quickly a difference of less than a hundredth affects the equation to bring completely different results. Taking the difference between the first and second values leads to a histogram that approximates a normal distribution.

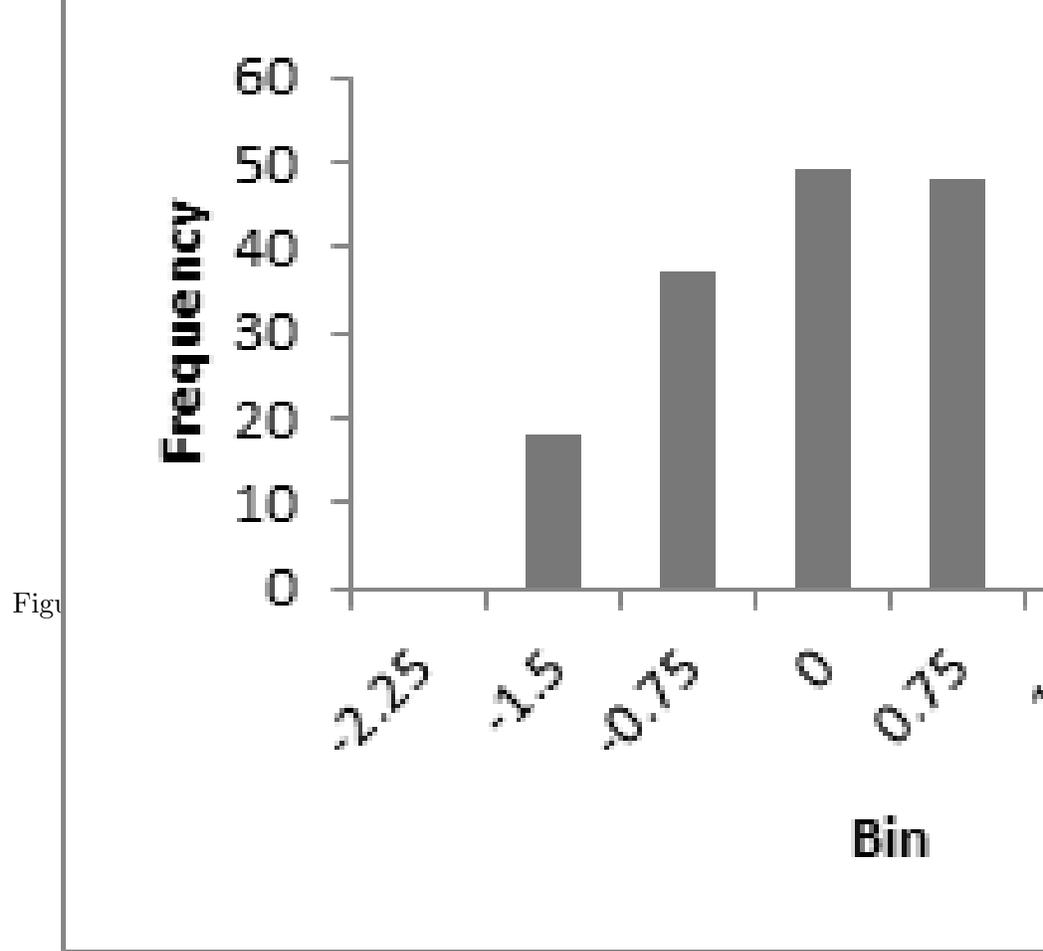


Figure 4: Histogram of differences, 200 iterations

An illustration of chaos is the chaos game. Beginning with any regular polygon, a random point is selected inside of it. Each new point is placed the same fraction of the distance between the previous point and a randomly chosen vertex. This will often approximate a fractal. For example, a triangle with points placed halfway between the previous point and a random vertex after hundreds of iterations will become a fractal commonly called Sierpinski's triangle. This particular fractal can also be generated by starting with an equilateral triangle, connecting the midpoints of the sides to create four identical smaller triangles and then removing the middle piece.

This is done repeatedly :



Figure 5: [7]

The chaos game generation of Sierpinski's triangle works because it is random. If each vertex was used in an orderly manner, ABCABCABC, the result would be (after transition states removed) the same three points being visited endlessly in succession. The chaos game works because no matter where the initial point is (for example, point D in the figure) and no matter which vertex is selected, (A, B or C), the next point will end up in one of the shaded regions. In my diagram, selecting A, B, or C will land the next point at, respectively, points x, y, or z. Since Sierpinski's triangle is entirely composed of miniatures of itself, points will continue to avoid the removed middle triangles. It is important to note that the first hundred iterations are removed since the initial point can be selected in an empty space.

Fractals are self similar on all scales, that is a portion of the fractal looks the same as the whole [8]. Examples of fractals in nature include plants such as ferns and broccoli, mountain ranges and coastlines but these are not perfect fractals. A close up view of a fern will show up as individual cells which do not resemble the whole [9]. Although not perfect representations

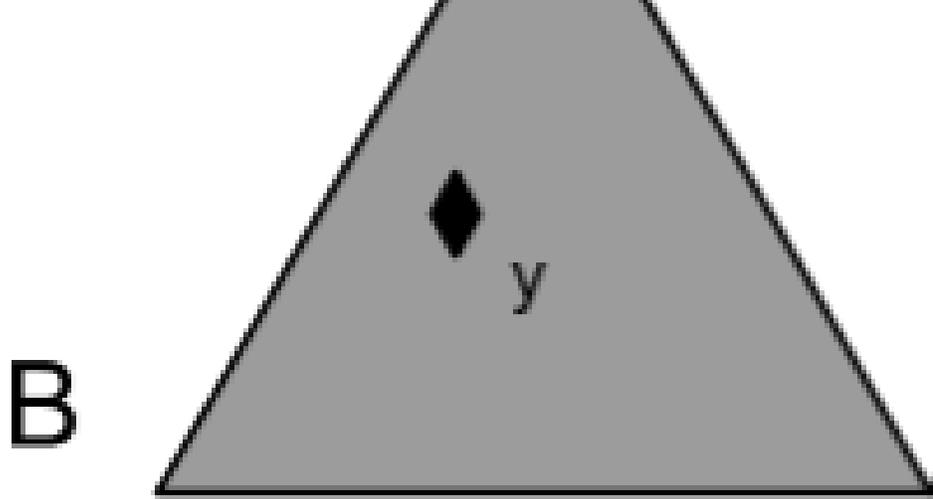


Figure 6

of nature, mathematical fractals are useful for approximating nature.

Another property of fractals is that their self-similarity dimension is greater than the topological dimension. The self-similarity dimension of a shape is defined as the number of copies of the original shape will fit inside a scaled up version of it, or

$$\text{number of small copies} = (\text{magnification factor})^{\text{Dimension}[9]} \quad (4)$$

solving for dimension gives us,

$$\frac{\log(\text{number of small copies})}{\log(\text{magnification factor})} = \text{Dimension}[9] \quad (5)$$

For example, a line, magnified by three will result in a line that fits three of the original line, thus $\log 3 / \log 3 = 1$. Similarly, a square magnified by three, both length and width, results in a square that fits nine of the original, $\log 9 / \log 3 = 2$. Finally, a cube, once three times as large will be able to fit twenty-seven of the original cube, thus $\log 27 / \log 3 = 3$ a cube is three dimensional. Looking at Sierpinski's triangle for the initial step and the first iteration (results would be the same no matter which consecutive iterations

used), the first iteration has three of the original triangle within it, however they need to be scaled u

[9].

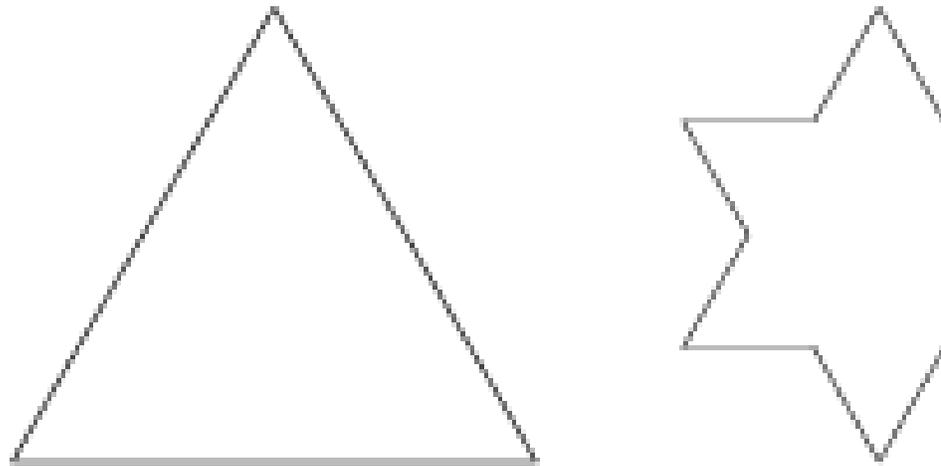


Figure 7: First four iterations of the Koch Snowflake [10]

The Koch curve is another fractal, generated by taking the middle third of a line segment and raise a triangle over it, removing the base that would be a part of the original line segment. [11]. Similarly, the Koch snowflake uses the same generator but begins with a triangle and the process is done on each of the three line segments. The fractal dimension of the Koch snowflake is $\log 4 / \log 3 \approx 1.261859$ [4].

The topological dimension is the usual definition of the dimension of an object; a point is zero dimensional, line is one, plane is two, and a cube is three. The Sierpinski triangle is one dimensional since as more triangles are removed from the interior, what remains is the lines that define the bounds of each triangle [9]. Similarly, the Koch snowflake is also one dimensional since it is also composed of lines. The Cantor set, begins with a line segment and

each iteration removes the middle third of each line segment, is composed of points and thus has a topological dimension of zero and a fractal dimension of ≈ 0.6309 [9].



Figure 8: Cantor set [12]

Fractal dimensions can also be determined by using the box counting method. The shape is covered in boxes of side length s . The number of tiles needed to cover the shape is $N(s)$. A table is created using different sizes of boxes. The table is used to solve

$$N(s) = k \frac{1}{s}^D \quad (7)$$

for D , the dimension, and k , a constant. This method has the advantage of not relying on exact miniatures of the whole that the self similarity dimension requires. The disadvantage is that some shapes can not be covered by boxes without overhang, such as a circle. The number of boxes can also depend on how the boxes are placed. Taking the log of the equation above gives,

$$\log(N(s)) = \log\left(k \left(\frac{1}{s}\right)^D\right) \quad (8)$$

using properties of logs to simplify,

$$\log(N(s)) = \log(k) + D\log\left(\frac{1}{s}\right)[9] \quad (9)$$

Substituting in y for $\log(N(s))$ and x for $\log\left(\frac{1}{s}\right)$,

$$y = k + Dx \quad (10)$$

shows it is a linear relationship. The box counting dimension can then be approximated by plotting x and y and determining the slope of the resulting line. [9].

Fractals having dimensions between one and two means they have properties of both one and two dimensional objects and similarly for fractals with a self similar dimension between zero and one. For example, the Koch snowflake resides in the two dimensional plane, but is composed of a series of one dimensional lines [9]. The fractal dimension measures the rate of addition of structural detail with increasing magnification serving as a quantifier of complexity [13]. It is a measure of the crinkliness, or edginess of the fractal. The Koch snowflake is composed of one dimensional lines but due to its crinkliness, its perimeter is infinite as the number of iterations approaches infinity [4].

One use of fractals is to create scenery for movies. Examples include the forest of Endor in *Star Wars: Return of the Jedi* and mountains in *Star Trek II: The Wrath of Khan* [14]. Irregularity and randomness are introduced to make the normally symmetric fractals appear more similar to the imperfect fractals created in nature. An irregular Sierpinski triangle is created by removing a non equilateral triangle from each triangle instead of



an equilateral triangle. A
a coin to decide for each
These fractals have stati
fractals are composed of
the whole. [9]

Figure 9: an example of a random Koch curve [15]

A second use of fractals is in identifying the type and malignancy of cancer cells. The box counting dimension of the cells are used to identify the type of cells and assign a grade of tumor malignancy solely based on fractal dimension analysis. [?] The study examined two sister cell lines, PatuT and PatuS, and was correctly able to identify the PatuT cells 97% of the time. The alternative method of tumor marking for pancreatic adenocarcinoma CA199 has a sensitivity of 85%. The authors note that the fractal dimension of the cell quantifies the tumor progression, which is associated with the amount of chaos within the cell. [?]. Randomness and order are not mutually exclusive. Sierpinski's triangle, an ordered, symmetrical figure is generated by a random process.

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